解析学演習（第１回）

学籍番号　　　　　　　　氏名

自作問題.　次の重積分の値を求めよ.

(1) $\displaystyle\iint_D x^2+4xy-3y^2\,dxdy$
$D=\{\ (x,y)\mid -1\leqq x\leqq 2,\ 0\leqq y\leqq 3\ \}$

$$\int_{-1}^{2}\int_{0}^{3}(x^2+4xy-3y^2)\,dy\,dx$$
$$=\int_{-1}^{2}\Big[x^2y+2xy^2-y^3\Big]_0^3dx$$
$$=\int_{-1}^{2}(3x^2+18x-27)\,dx$$
$$=\Big[x^3+9x^2-27x\Big]_{-1}^{2}$$
$$=9+9\cdot 3-27\cdot 3$$
$$=\underline{-45}$$

(2) $\displaystyle\iint_D e^{x-y}\,dxdy$
$D:=\{\ (x,y)\mid 0\leqq x\leqq 1,\ -1\leqq y\leqq 1\ \}$

$$\int_0^1\int_{-1}^{1}e^{x-y}\,dy\,dx$$
$$=\int_0^1 e^x\Big[-e^{-y}\Big]_{-1}^{1}dx$$
$$=\left(e-\frac{1}{e}\right)\Big[e^x\Big]_0^1$$
$$=\left(e-\frac{1}{e}\right)(e-1)=\underline{\frac{1}{e}(e-1)^2(e+1)}$$

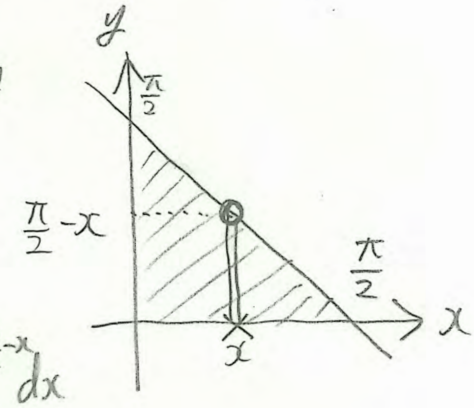
自作問題.　次の重積分の値を求めよ (括弧内は D を表す).

(1) $\displaystyle\iint_D x^2y\,dxdy$
$D=\{\ (x,y)\mid 0\leqq x\leqq 1,\ 0\leqq y\leqq \sqrt{1-x^2}\ \}$

$$\int_0^1\int_0^{\sqrt{1-x^2}}x^2y\,dy\,dx$$
$$=\int_0^1 x^2\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}}dx$$
$$=\frac{1}{2}\int_0^1 x^2(1-x^2)\,dx$$
$$=\frac{1}{2}\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1$$
$$=\underline{\frac{1}{15}}$$

(2) $\displaystyle\iint_D \sin(x+y)\,dxdy$
$D=\left\{\ (x,y)\ \middle|\ x,y\geqq 0,\ x+y\leqq\frac{\pi}{2}\ \right\}$

$$\iint_D \sin(x+y)\,dxdy$$
$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}-x}\sin(y+x)\,dy\,dx$$



$$=\int_0^{\frac{\pi}{2}}\Big[-\cos(y+x)\Big]_0^{\frac{\pi}{2}-x}dx$$
$$=\int_0^{\frac{\pi}{2}}\cos x\,dx=\Big[\sin x\Big]_0^{\frac{\pi}{2}}$$
$$=\underline{1}$$

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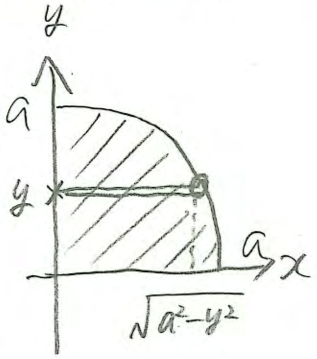
Text p.200, 1.　次の重積分の値を求めよ.

(3) $\displaystyle\iint_D xy\,dxdy$
$D = \{\ (x, y) \mid x \geqq 0,\ y \geqq 0,\ x^2 + y^2 \leqq a^2\ \}$



$$\iint_D xy\,dxdy$$

$$= \int_0^a y \int_0^{\sqrt{a^2-y^2}} x\,dx\,dy$$

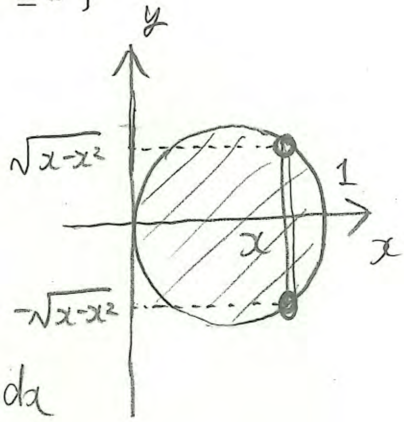
$$= \int_0^a y \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a y(a^2-y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{a^4}{8}$$

(4) $\displaystyle\iint_D \sqrt{x}\,dxdy$
$D = \{\ (x, y) \mid x^2 + y^2 \leqq x\ \}$



$$\iint_D \sqrt{x}\,dxdy$$

$$= \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \sqrt{x}\,dy\,dx$$

$$= \int_0^1 \sqrt{x}\,\Big[\,y\,\Big]_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} dx$$

$$= 2\int_0^1 x\sqrt{1-x}\,dx$$

$$= 2\left(\int_0^1 (1-x)^{\frac{1}{2}}dx - \int_0^1 (1-x)^{\frac{3}{2}}dx\right)$$

$$= 2\left[-\frac{2}{3}(1-x)^{\frac{3}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}}\right]_0^1$$

$$= 2\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{8}{15}$$

(6) $\displaystyle\iint_D \sin\frac{\pi y}{\sqrt{x}}\,dxdy$
$D = \{\ (x, y) \mid y^2 \leqq x,\ 1 \leqq x \leqq 2\ \}$

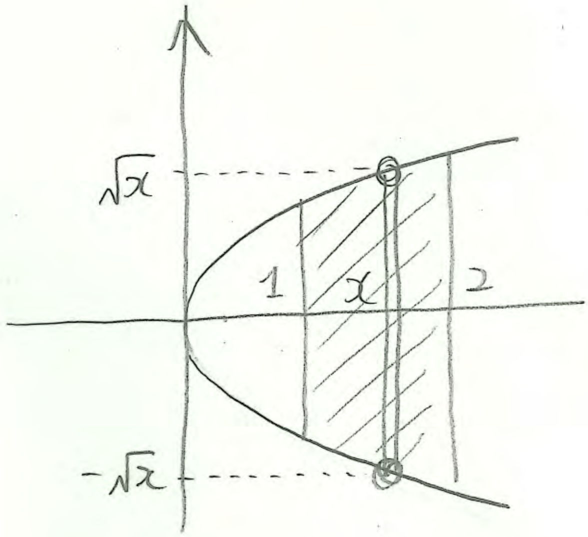
$$\iint_D \sin\frac{\pi y}{\sqrt{x}}\,dx\,dy$$

$$= \int_1^2 \int_{-\sqrt{x}}^{\sqrt{x}} \sin\frac{\pi y}{\sqrt{x}}\,dy\,dx$$

$$= \int_1^2 \left[-\frac{\sqrt{x}}{\pi}\cos\frac{\pi y}{\sqrt{x}}\right]_{-\sqrt{x}}^{\sqrt{x}} dx$$

$$= \int_1^2 -\frac{\sqrt{x}}{\pi}\left(\cos\pi - \cos(-\pi)\right) dx$$

$$= \int_1^2 0\,dx = 0$$



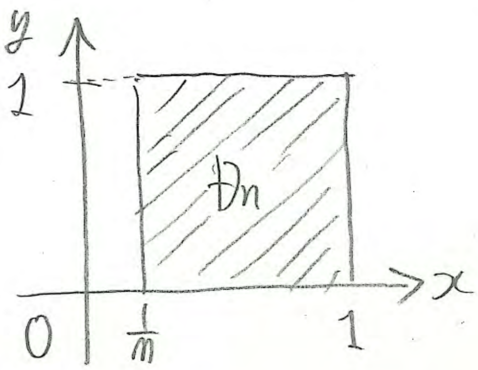
解析学（第2回）

学籍番号　　　　　　　　　氏名

次の広義重積分を計算せよ．

1. $\displaystyle\iint_D \frac{dx\,dy}{(x+y)^{\frac{3}{2}}}$

$$D=\left\{ (x,y) \,\middle|\, \begin{array}{l} 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1, \\ (x,y) \neq (0,0) \end{array} \right\}$$



$$D_n=\left\{(x,y) \,\middle|\, \begin{array}{l} \frac{1}{n} \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array}\right\},\ n\in\mathbb{N}.$$

とおくと

$$\iint_{D_n} \frac{dxdy}{(x+y)^{\frac{3}{2}}} = \int_{\frac{1}{n}}^{1}\int_0^1 (x+y)^{-\frac{3}{2}}dy\,dx$$

$$= \int_{\frac{1}{n}}^{1}\left[-2(x+y)^{-\frac{1}{2}}\right]_{y=0}^{y=1} dx$$

$$= -2\int_{\frac{1}{n}}^{1}\left((x+1)^{-\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx$$

$$= -2\left[2(x+1)^{\frac{1}{2}} - 2x^{\frac{1}{2}}\right]_{\frac{1}{n}}^{1}$$

$$= -4\left(2^{\frac{1}{2}} - 1 - \left(\frac{1}{n}+1\right)^{\frac{1}{2}} + \left(\frac{1}{n}\right)^{\frac{1}{2}}\right)$$

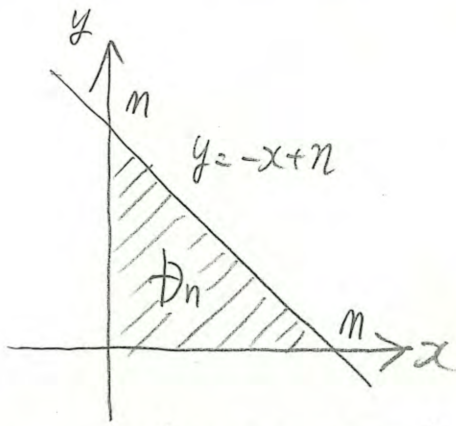
$$\to -4(\sqrt{2}-2) = 4(2-\sqrt{2})$$

よって

$$\iint_D \frac{dxdy}{(x+y)^{\frac{3}{2}}} = \underline{4(2-\sqrt{2})}$$

2. $\displaystyle\iint_D \frac{dx\,dy}{(1+x+y)^{\alpha}}$ $(\alpha > 2)$

$$D=\{\ (x,y) \mid x \geqq 0,\ y \geqq 0\ \}$$



$$D_n=\left\{(x,y) \,\middle|\, \begin{array}{l} x \geq 0,\ y \geq 0 \\ x+y \leq n \end{array}\right\}$$

とおくと

$$\iint_{D_n} \frac{dxdy}{(1+x+y)^{\alpha}}$$

$$= \int_0^n \int_0^{-x+n} (1+x+y)^{-\alpha}\,dy\,dx$$

$$= \int_0^n \left[\frac{1}{1-\alpha}(1+x+y)^{1-\alpha}\right]_{y=0}^{y=-x+n} dx$$

$$= \frac{1}{1-\alpha}\left((1+n)^{1-\alpha}\cdot n - \left[\frac{1}{2-\alpha}(1+x)^{2-\alpha}\right]_0^n\right)$$

$$= \frac{1}{1-\alpha}\left((1+n)^{2-\alpha} - (1+n)^{1-\alpha} - \frac{(1+n)^{2-\alpha}}{2-\alpha} + \frac{1}{2-\alpha}\right)$$

$$= \frac{1}{(\alpha-1)(\alpha-2)}\left(\frac{\alpha-1}{(1+n)^{\alpha-2}} + \frac{\alpha-2}{(1+n)^{\alpha-1}} + 1\right)$$

$$\to \frac{1}{(\alpha-1)(\alpha-2)}$$

よって

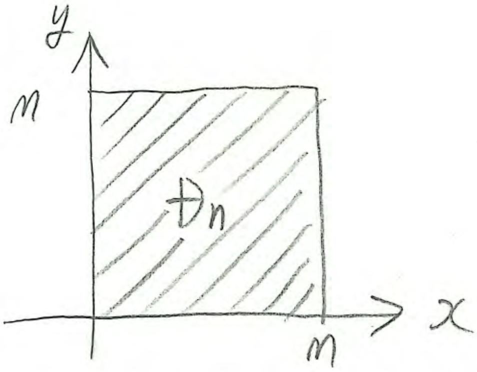
$$\iint_D \frac{dxdy}{(1+x+y)^{\alpha}} = \underline{\frac{1}{(\alpha-1)(\alpha-2)}}$$

(裏面に続く)

次の広義重積分を計算せよ．

3. $\displaystyle\iint_D ye^{-(x+y)}dx\,dy$

$D = \{\ (x,y) \mid x \geqq 0,\ y \geqq 0\ \}$



$$D_n = \left\{ (x,y) \,\middle|\, \begin{array}{l} 0 \leq x \leq n \\ 0 \leq y \leq n \end{array} \right\},\ n \in \mathbb{N}$$

とおくと

$$\iint_{D_n} y\, e^{-(x+y)}\, dx dy$$

$$= \int_0^n \int_0^n y e^{-(x+y)}\, dx\, dy$$

$$= \left(\int_0^n e^{-x} dx\right)\left(\int_0^n y e^{-y} dy\right)$$

$$= \Big[-e^{-x}\Big]_0^n \Big[-(y+1)e^{-y}\Big]_0^n$$

$$= \left(1 - \frac{1}{e^n}\right)\left(1 - \frac{n+1}{e^n}\right)$$

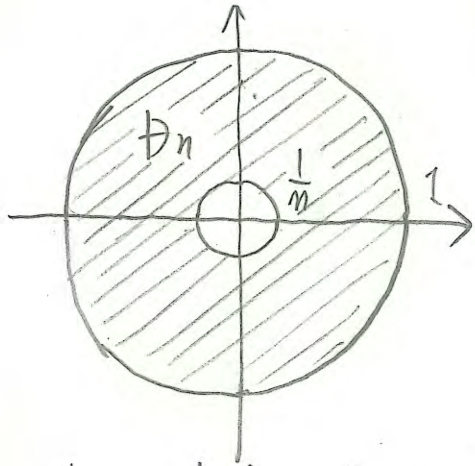
$$\longrightarrow 1 \quad \text{as} \quad n \to \infty$$

よって

$$\iint_D y e^{-(x+y)}\, dx dy = 1$$

4. $\displaystyle\iint_D \log(x^2+y^2)\ dx\,dy$

$D := \{\ (x,y) \mid 0 < x^2 + y^2 \leqq 1\ \}$



$$D_n = \left\{ (x,y) \,\middle|\, \frac{1}{n^2} \leq x^2 + y^2 \leq 1 \right\},\ n \in \mathbb{N}$$

とおいて極座標変換：

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \left(\begin{array}{l} \frac{1}{n} \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}\right)$$

を用いて計算すると

$$\iint_{D_n} \log(x^2+y^2)\, dx dy$$

$$= \int_{\frac{1}{n}}^1 \int_0^{2\pi} (\log r^2) \cdot r\, d\theta\, dr$$

$$= \pi \int_{\frac{1}{n}}^1 2r \log r^2\, dr$$

$$= \pi \Big[r^2 \log r^2 - r^2 \Big]_{\frac{1}{n}}^1$$

$$= \pi \left(-1 - \frac{1}{n^2} \log \frac{1}{n^2} + \frac{1}{n^2} \right)$$

$$= -\pi + \frac{2}{n} \frac{\log n}{n} + \frac{1}{n^2}$$

$$\longrightarrow -\pi \quad \text{as} \quad n \to \infty$$

よって $\displaystyle\iint_D \log(x^2+y^2)\, dx dy = -\pi$

解析学（第３回）

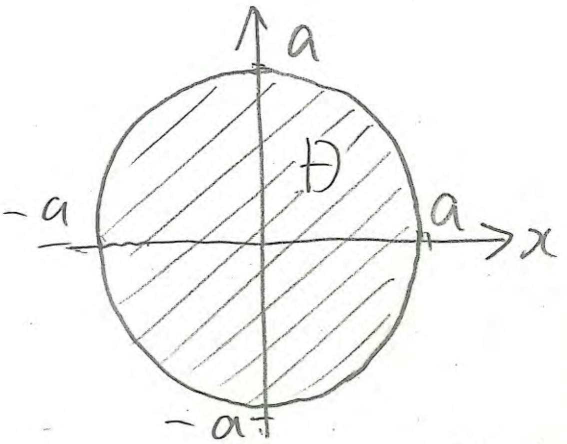
学籍番号　　　　　　　　氏名

次の重積分を計算せよ．

1. $I = \iint_D \sqrt{a^2 - x^2 - y^2}^{\,3}\, dxdy$

$D = \{\ (x, y) \mid x^2 + y^2 \leqq a^2\ \}$

($a > 0$: 定数)



図より. 領域 D は, 変数変換

$$\begin{cases} x = \varphi(r,\theta) = r\cos\theta \\ y = \psi(r,\theta) = r\sin\theta \end{cases}$$

によって

$$\Omega = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

へ変換される

$\dfrac{\partial(\varphi,\psi)}{\partial(r,\theta)} = r$ であるから.

$$I = \int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2}^{\,3} \cdot r\, d\theta\, dr$$

$$= 2\pi \int_0^a r\sqrt{a^2 - r^2}^{\,3}\, dr$$

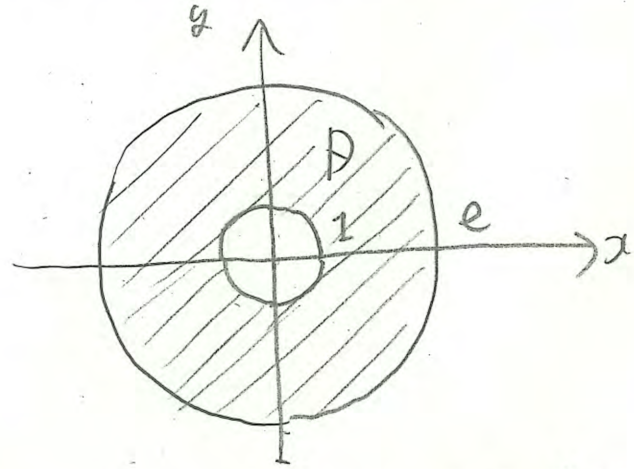
$$= -\pi \int_0^a (-2r)(a^2 - r^2)^{\frac{3}{2}}\, dr$$

$$= -\pi \left[\frac{2}{5}(a^2 - r^2)^{\frac{5}{2}} \right]_0^a$$

$$= \underline{\frac{2}{5}\pi a^5}$$

2. $I = \iint_D \log(x^2 + y^2)\, dxdy$

$D := \{\ (x, y) \mid 1 \leqq x^2 + y^2 \leqq e^2\ \}$



図より. 領域 D は, 変数変換

$$\begin{cases} x = \varphi(r,\theta) = r\cos\theta \\ y = \psi(r,\theta) = r\sin\theta \end{cases}$$

によって

$$\Omega = \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 1 \leq r \leq e \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

へ変換される

$\dfrac{\partial(\varphi,\psi)}{\partial(r,\theta)} = r$ であるから.

$$I = \int_1^e \int_0^{2\pi} \log(r^2\cos^2\theta + r^2\sin^2\theta)\, r\, d\theta\, dr$$

$$= 2\pi \int_1^e r \log r^2\, dr$$

$$= \pi \left[r^2 \log r^2 - r^2 \right]_1^e$$

$$= \pi(2e^2 - e^2 + 1)$$

$$= \underline{\pi(e^2 + 1)}$$

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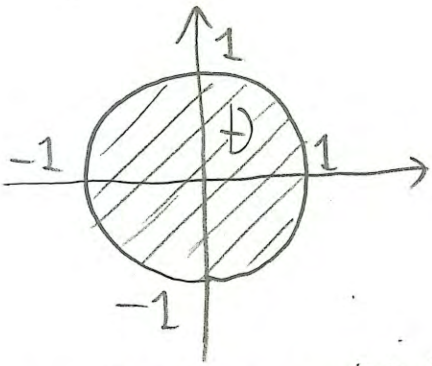
次の重積分を計算せよ.

3. $I = \iint_D (2x^2 + 3y^2)\,dxdy$

$D := \{\ (x,y) \mid x^2 + y^2 \leqq 1\ \}$



図より，領域 D は，変数変換

$$\begin{cases} x = \varphi(r,\theta) = r\cos\theta \\ y = \psi(r,\theta) = r\sin\theta \end{cases}$$

によって

$$\Omega = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

へ変換される

$\dfrac{\partial(\varphi,\psi)}{\partial(r,\theta)} = r$ であるから

$$I = \int_0^1 \int_0^{2\pi} (2r^2\cos^2\theta + 3r^2\sin^2\theta)\cdot r\, d\theta\, dr$$

$$= \left(\int_0^1 r^3\, dr\right) \times$$

$$\times \left(\int_0^{2\pi} \left((1+\cos 2\theta) + \frac{3}{2}(1-\cos 2\theta)\right) d\theta\right)$$

$$= \left[\frac{r^4}{4}\right]_0^1 \int_0^{2\pi} \left(\frac{5}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{4}\cdot\frac{5}{2}\cdot 2\pi = \frac{5}{4}\pi$$

4. $I = \iint_D (x^2 - y^2) e^{-x-y}\,dxdy$

$D := \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \leqq x+y \leqq 1, \\ 0 \leqq x-y \leqq 1 \end{array} \right\}$

$u = x+y,\ v = x-y$ とおくと

(x,y) から (u,v) への変換は

$$\begin{cases} x = \varphi(u,v) = \frac{1}{2}(u+v) \\ y = \psi(u,v) = \frac{1}{2}(u-v) \end{cases}$$

となり 領域 D は，

$$\Omega = \left\{ (u,v) \;\middle|\; \begin{array}{l} 0 \leq u \leq 1 \\ 0 \leq v \leq 1 \end{array} \right\}$$

へ変換される

$$\frac{\partial(\varphi,\psi)}{\partial(u,v)} = \det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} = -\frac{1}{2}$$

であるので

$$I = \int_0^1 \int_0^1 uv\, e^{-u} \left|-\frac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}\left(\int_0^1 v\, dv\right)\left(\int_0^1 u\, e^{-u} du\right)$$

$$= \frac{1}{4}\left(\left[-ue^{u}\right]_0^1 + \int_0^1 e^{-u} du\right)$$

$$= \frac{1}{4}\left(-\frac{1}{e} + \left[-e^{-u}\right]_0^1\right)$$

$$= \frac{1}{4}\left(-\frac{2}{e} + 1\right)$$

$$= \frac{1}{4} - \frac{1}{2e}$$

解析学（第４回）

学籍番号　　　　　　　　　　氏名

問題. つぎのデータに対して $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ を計算せよ.

1. $\boldsymbol{F}(x,y) = (y^2, -x^2)$,
$C: \boldsymbol{r}(t) = (t, 4t)\ (0 \leqq t \leqq 1)$

$\mathbb{F}(\mathbb{r}(t)) = (16t^2, -t^2)$

$\mathbb{r}'(t) = (1, 4)$

$\int_C \mathbb{F}(\mathbb{r}) \cdot d\mathbb{r}$

$= \int_0^1 (16t^2, -t^2) \cdot (1, 4)\, dt$

$= 12 \int_0^1 t^2 dt$

$= 12 \left[\frac{t^3}{3} \right]_0^1$

$= 4$

2. $\boldsymbol{F}(x,y) = (2y, x)$,
$C: \boldsymbol{r}(t) = (\cos t, \sin t)\ (0 \leqq t \leqq 2\pi)$

$\mathbb{F}(\mathbb{r}(t)) = (2\sin t, \cos t)$

$\mathbb{r}'(t) = (-\sin t, \cos t)$

$\int_C \mathbb{F}(\mathbb{r}) \cdot d\mathbb{r}$

$= \int_0^{2\pi} (2\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$

$= \int_0^{2\pi} (-2\sin^2 t + \cos^2 t)\, dt$

$= \int_0^{2\pi} \left(-(1-\cos 2t) + \frac{1}{2}(1+\cos 2t) \right) dt$

$= -\frac{1}{2} \cdot 2\pi = -\pi$

(裏面に続く)

問題. つぎのデータに対して $\int_C \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}$ を計算せよ.

3. $\boldsymbol{F}(x,y) = (e^x, e^{-y})$,

C: $\boldsymbol{r}(t) = (t, t^2)$ $(0 \leqq t \leqq 1)$

$$\mathbb{F}(\mathbb{r}(t)) = (e^t, e^{-t^2})$$

$$\mathbb{r}'(t) = (1, 2t)$$

$$\int_C \mathbb{F}(\mathbb{r})\cdot d\mathbb{r}$$

$$= \int_0^1 (e^t, e^{-t^2})\cdot(1, 2t)\,dt$$

$$= \int_0^1 (e^t + 2te^{-t^2})\,dt$$

$$= \left[e^t - e^{-t^2}\right]_0^1$$

$$= (e-1) - (e^{-1}-1)$$

$$= e - \frac{1}{e}$$

4. $\boldsymbol{F}(x,y) = \left(\frac{e^x + e^{-x}}{2}, \frac{e^y - e^{-y}}{2}\right)$,

C: $\boldsymbol{r}(t) = (t, t^2)$ $(0 \leqq t \leqq 2)$

$$\mathbb{F}(\mathbb{r}(t)) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^{t^2} - e^{-t^2}}{2}\right)$$

$$\mathbb{r}'(t) = (1, 2t)$$

$$\int_C \mathbb{F}(\mathbb{r})\cdot d\mathbb{r}$$

$$= \int_0^2 \left(\frac{e^t + e^{-t}}{2}, \frac{e^{t^2} - e^{-t^2}}{2}\right)\cdot(1, 2t)\,dt$$

$$= \frac{1}{2}\int_0^2 (e^t + e^{-t})\,dt$$

$$+ \frac{1}{2}\int_0^2 (2te^{t^2} - 2te^{-t^2})\,dt$$

$$= \frac{1}{2}\left[e^t - e^{-t}\right]_0^2 + \frac{1}{2}\left[e^{t^2} + e^{-t^2}\right]_0^2$$

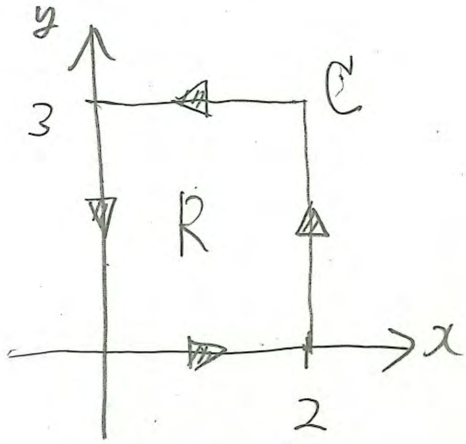
$$= \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) + \frac{1}{2}\left(e^4 + \frac{1}{e^4} - 2\right)$$

解析学（第 5 回）

学籍番号　　　　　　　　氏名

問題. 領域 R の境界曲線 C 上の反時計回りの線積分 $\int_C \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}$ をグリーンの定理を用いて計算せよ.

1. $\boldsymbol{F} = [x^2e^y, y^2e^x]$, C : 頂点 $(0,0)$, $(2,0)$, $(2,3)$, $(0,3)$ の長方形領域



グリーンの定理から.

$$\int_C \mathbb{F}(\mathbb{r})\cdot d\mathbb{r}$$

$$= \iint_R \left(\frac{\partial}{\partial x}(y^2e^x) - \frac{\partial}{\partial y}(x^2e^y)\right)dx\,dy$$

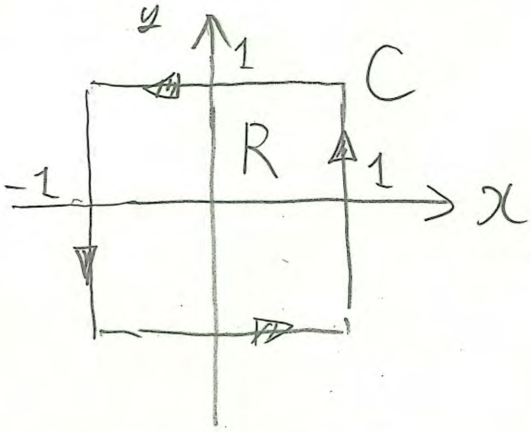
$$= \int_0^3\int_0^2 (y^2e^x - x^2e^y)\,dx\,dy$$

$$= \left(\int_0^3 y^2\,dy\right)\left(\int_0^2 e^x\,dx\right) - \left(\int_0^2 x^2\,dx\right)\left(\int_0^3 e^y\,dy\right)$$

$$= \left[\frac{y^3}{3}\right]_0^3 \left[e^x\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2 \left[e^y\right]_0^3$$

$$= 9(e^2-1) - \frac{8}{3}(e^3-1)$$

2. $\boldsymbol{F} = [3y^2, x-y^4]$, C : 頂点 $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ の長方形領域



グリーンの定理から

$$\int_C \mathbb{F}(\mathbb{r})\cdot d\mathbb{r}$$

$$= \iint_D \left(\frac{\partial}{\partial x}(x-y^4) - \frac{\partial}{\partial y}(3y^2)\right)dx\,dy$$

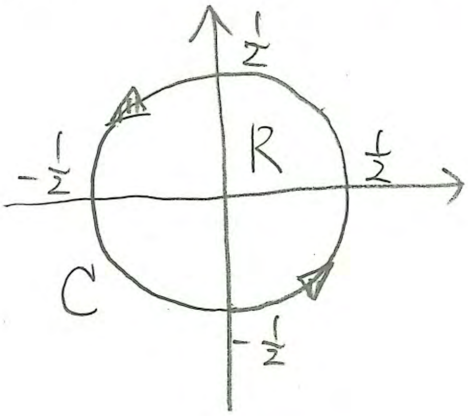
$$= \int_{-1}^1\int_{-1}^1 (1-6y)\,dx\,dy$$

$$= 2\left[y-3y^2\right]_{-1}^1$$

$$= 4$$

問題. 領域 R の境界曲線 C 上の反時計回りの線積分 $\int_C \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}$ をグリーンの定理を用いて計算せよ.

3. $\boldsymbol{F}=[y,-x]$, C : 円 $x^2+y^2=\frac{1}{4}$



グリーンの定理から

$$\int_C \mathbb{F}(\mathbb{r})\cdot d\mathbb{r}$$

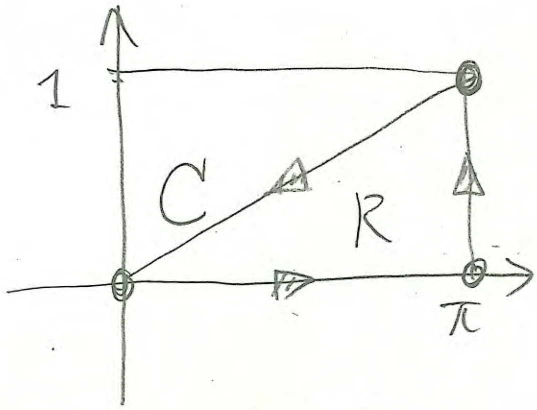
$$= \iint_D \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right) dx\,dy$$

$$= -2\iint_D 1\,dx\,dy$$

$$= -2\times\pi\times\left(\frac{1}{2}\right)^2$$

$$= -\frac{\pi}{2}$$

4. $\boldsymbol{F}=[\sin y,\cos x]$, C : 頂点 $(0,0)$, $(\pi,0)$, $(\pi,1)$ の 3 角形領域



グリーンの定理から

$$\int_C \mathbb{F}(\mathbb{r})\cdot d\mathbb{r}$$

$$= \iint_D \left(\frac{\partial}{\partial x}\cos x - \frac{\partial}{\partial y}\sin y\right) dx\,dy$$

$$= \int_0^{\pi}\int_0^{\frac{x}{\pi}} (-\sin x - \cos y)\,dy\,dx$$

$$= \int_0^{\pi}\left[-y\sin x - \sin y\right]_0^{\frac{x}{\pi}} dx$$

$$= \int_0^{\pi}\left(-\frac{x}{\pi}\sin x - \sin\frac{x}{\pi}\right) dx$$

$$= -\frac{1}{\pi}\left(\left[-x\cos x\right]_0^{\pi} + \int_0^{\pi}\cos x\,dx\right) + \left[\pi\cos\frac{x}{\pi}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\cdot\pi + \pi\cos 1 - \pi$$

$$= \pi\cos 1 - \pi - 1.$$